Initial Decoherence and Disentanglement of Open Two-Qubit Systems

Xian-Ting Liang

Department of Physics and Institute of Modern Physics, Ningbo University, Ningbo, 315211, China

Abstract

In this paper we investigate a open two-qubit model whose dynamics is not exactly solvable. When the initial state is the maximum entangled state, as the exactly solvable open two-qubit model [D. Tolkunov and V. Privman, Phys. Rev. A 71, 060308(R) (2005)], the decay of entanglement of formation of the model, expressed by concurrence is also governed by the product of suppression factors describing decoherence of the subsystems (qubits). However, if the initial state is not the maximum entangled state, its concurrence will decrease faster than the product of the suppression factors describing decoherence of the qubits.

Keywords: Concurrence; short-time approximation; decoherence.

PACS numbers: 03.67.Mn, 03.65.Yz, 03.65.Ud

I. INTRODUCTION

Coherence and entanglement are two remarkable features of quantum systems, and they are also dominating reasons why quantum computation and quantum communication differ from the classical ones [1]. For example, quantum coherence can lead to natural parallel computations which can enhance efficiency for solving some complex problems by using effective quantum algorithms. Quantum bit (qubit) is a key block for building quantum computers. However, the interactions of qubits with their environment will induce loss of the coherence, decoherence. Decoherence is considered a central impediment to fabricate quantum computers. In quantum computation and quantum communication, the entanglement, nonlocal correlation between the quantum subsystems is also needed. The entanglement is recognized as an important physical resource in quantum information transmission and processing. Many protocols in quantum communication and quantum computation are based on the entangled states. It is also shown that couplings of the quantum systems and its subsystems to their environment will result in the loss of the entanglement, and this loss cannot be restored by local operations and classical communications [2, 3]. Both the decoherence and the loss of the entanglement may result from the interactions of the quantum systems with their environment. Then, what relations are there between the two quantities? In this paper we will investigate the relationship of the decoherence and the loss of entanglement for a non-exactly solvable two-qubit model. As Ref.[4, 5], let us refer to two subsystems, $S^{(1)}$ and $S^{(2)}$, of the combined system, S. The evolutions of the coherence of the subsystems $S^{(1)}$ and $S^{(2)}$ in their bathes can be described by suppression factors, $0 \leq \delta^{(1,2)}(t) \leq 1$. On the other hand, the evolutions of the entanglement between the subsystems $S^{(1)}$ and $S^{(2)}$ can be described by entanglement of formation [6] expressed by concurrence [15]. In large times the decay of the coherent terms are exponential, then the decay rate of the whole system is the summation of the rates of the subsystems [7, 8, 18]. Yu and Eberly [9] found that for a exactly solvable model in which the decoherence is caused by pure dephasing process, the concurrence decays faster than the quantum dephasing of an individual qubit. A more physical model, two entangled atoms in pure vacuum noise, is investigated recently by the same authors [10]. They found that the disentanglement time is shorter than the usual spontaneous lifetime. V. Privman *et al.* [4, 5] investigated another dynamically solvable model. They shown that the *decay of concurrence* of the model is governed by the product of suppression factors describing decoherence of the subsystems (qubits) in a short time. It is interesting that whether or not the relationship of the concurrence and the suppression factors describing the decoherence can be held for other open quantum systems. In this paper we shall investigate a different open two-qubit model whose dynamics is not exactly solvable. By using a short-time approximation we can obtain the evolution of reduced density matrix of the system. It will be shown that as the initial state of the two-qubit system is the maximum entangled states the decay of concurrence, namely the decay of the degree of entanglement of formation can also be governed by the product of suppression factors describing decoherence of the subsystems (qubits) in a short time. If the initial state is not the maximum entangled state, the concurrence will decrease faster than the product of the suppression factors describing the coherence of the qubits does.

II. DECOHERENCE AND THE LOSS OF ENTANGLEMENT

Suppose the open two-qubit system has Hamiltonian

$$H = \sum_{r=1}^{2} \left(H_s^r + H_B^r + H_I^r \right), \tag{1}$$

where

$$\begin{aligned}
H_s^r &= -\frac{1}{2}E_J^r\sigma_x^r, \\
H_B^r &= \sum_k M_k^r = \sum_k \omega_k^r b_k^{r\dagger} b_k^r, \\
H_I^r &= \sigma_z^r \sum_k \left(g_k^{r*} b_k^r + g_k^r b_k^{r\dagger}\right). \qquad (2)
\end{aligned}$$

Here, we use the subscripts or the superscripts $r = 1,2$ to label the qubits. H_s^r and H_B^r are the Hamiltonian of qubits and their bosonic bathes and H_I^r are the interactions of the qubits with their bathes [11]. Where we assume that each qubit interacts with its own bath. This is not a exactly solvable model. If we do not consider the interactions between the qubits the evolution operator of the combined system can be expressed as

$$U = U_1 \otimes U_2. \qquad (3)$$

The evolution operator of the single qubit is

$$U_r = e^{-iH^r\tau/\hbar} = e^{-i(H_s^r + H_I^r + H_B^r)t}, \qquad (4)$$

where $t = \tau/\hbar$. Due to non-conservation of H_s in this system, the evolution operator cannot be in a general way expressed as $e^{-iH_s^r t}e^{-i(H_I^r+H_B^r)t}$. But in the sort-time approximation, the operator can be approximately expressed as [12, 13]

$$U_r = e^{-iH_s^r t/2}e^{-i(H_I^r+H_B^r)t}e^{-iH_s^r t/2} + o(t^3). \qquad (5)$$

It has been proved that the expression is accurate enough as the time being short to the characteristic time [14]. So the elements of the reduced density matrix $\rho(t)$ in the basis of operator H_s can be expressed as

$$\begin{aligned}
\rho_{mn}^r = \; & \mathrm{Tr}_B \left\langle \varphi_m \right| e^{-iH_s^r t/2}e^{-i(H_I^r+H_B^r)t}e^{-iH_s^r t/2}R(0) \\
& e^{iH_s^r t/2}e^{i(H_I^r+H_B^r)t}e^{iH_s^r t/2}\left|\varphi_n\right\rangle. \qquad (6)
\end{aligned}$$

Here, we suppose the initial state of the system be $R^r(0) = \rho^r(0)\otimes\Theta^r$ where $\rho^r(0)$ and Θ^r are initial states of the qubit and its environment. The latter is the product of the bath modes density matrices θ_k^r. In the initial states, each bath mode k is assumed to be thermalized, namely,

$$\theta_k^r = \frac{e^{-\beta M_k^r}}{\mathrm{Tr}_k\left(e^{-\beta M_k^r}\right)}, \qquad (7)$$

where $\beta = 1/kT$, T is the temperature and k is the Boltzmann constant.

A. Decoherence and decay of coherence

In this subsection we shall investigate the decoherence of each qubit due to the interaction of the qubit with its own environment, a bath. Here, we denote the suppression factors describing the decoherence of each qubits with δ^r. The relationship of the suppression factors δ^r with the usual term decoherence D^r is $D^r = L(1-\delta^r)$, where L is the initial coherent terms of the density matrix, namely, the off-diagonal elements of the density matrix of the initial state for the $r-th$ qubit. $L = 1/2$ for the initial states with maximum coherent terms. We call $L\delta^r$ the *decay of coherence* of the $r-th$ qubit. Usually, the environment is assumed to be a large macroscopic the interaction with it leads to the thermal equilibrium at temperature T. In this case, Markovian type approximations can be used to quantified the decoherent process and it usually yields the exponential decay of the density matrix elements in the energy basis of the Hamiltonian H_s^r. Thus, the decay rates will be additive. In this time scale the measures of entropy and the first entropy can be used for quantifying the decoherence. But the decoherence of the qubit gate operations cannot be characterized by this methods because the time of the elementary quantum gate operation is much shorter than the thermal relaxation time. It has been shown that the norms $\|\sigma^r\|_\lambda$ is useful for describing the decoherence of the short-time evolution [14]. Here σ^r is the deviation operator defined as

$$\sigma^r(t) = \rho^r(t) - \rho_i^r(t), \qquad (8)$$

where $\rho^r(t)$ and $\rho_i^r(t)$ are density matrixes of the "real" evolution (with interaction) and the "ideal" one (without interaction) of the $r-th$ qubit. We can use the norm $\|\sigma^r\|_\lambda$ describing the decoherence and the norm is defined as [14]

$$\|\sigma^r\|_\lambda = \sup_{\varphi\neq 0}\left(\frac{\langle\varphi|\,\sigma^r\,|\varphi\rangle}{\langle\varphi\,|\varphi\rangle}\right)^{\frac{1}{2}}. \qquad (9)$$

For a qubit, the norm can be given by

$$\|\sigma^r\|_\lambda = \sqrt{\left|\sigma_{10}^r\right|^2 + \left|\sigma_{11}^r\right|^2}. \qquad (10)$$

It is shown that for a given system, the norm $\|\sigma^r\|_\lambda$ increase with time, reflecting the decoherence of the system. However, in general it is oscillated at the system's internal frequency. Thus, the decohering effect of the bath is better quantified by the maximal operator norm

$$D^r(t) = \sup_{\rho^r(0)}\left(\|\sigma^r(t,\rho^r(0))\|_\lambda\right). \qquad (11)$$

For our investigating model, by using Eqs.(6) and 11) we can obtain the decoherence of the qubit as

$$D^r(t) = \frac{1}{2}\left(1 - e^{-4G^r(t)}\right), \qquad (12)$$

where

$$G^r(t) = 2\sum_k \frac{|g_k^r|^2}{\omega_k^{r2}}\sin^2\frac{\omega_k^r t}{2}\coth\frac{\beta\omega_k^r}{2}. \qquad (13)$$

It is shown that the suppression factor describing the decoherence of the open qubit r is

$$\delta^r = e^{-4G^r(t)}. \qquad (14)$$

B. Loss of the entanglement and decay of the entanglement

Numerous measures of entanglement have been considered over the years. For quantum information content, the entanglement of formation has been a widely accepted measure of entanglement. The measure can measure the degree of entanglement not only for pure states but also for mixed states. The concurrence is a quantity monotonically related to the entanglement of formation. For a pure or mixed state, ρ_s, of two qubits, one can define the spin-flipped state,

$$\tilde{\rho}_s = (\sigma_y \otimes \sigma_y)\, \rho_s^* \,(\sigma_y \otimes \sigma_y)\,, \tag{15}$$

and the Hermitian matrix,

$$R(\rho_s) = \sqrt{\sqrt{\rho_s}\tilde{\rho}_s\sqrt{\rho_s}}, \tag{16}$$

with eigenvalues $\lambda_{i=1,2,3,4}$. Here, ρ_s^* denotes the complex conjugation of ρ in the standard basis and σ_y is one of the Pauli matrixes. The concurrence, C, is defined by [15]

$$C = \max\left\{0, 2\max_i \lambda_i - \sum_{j=1}^{4}\lambda_j\right\}. \tag{17}$$

From Eq.(6) and helped with operator-algebra techniques we can obtain the evolutions of the reduced density matrix as

$$\rho^r(t) = \begin{pmatrix} \rho_{00}^r(t) & \rho_{01}^r(t) \\ \rho_{10}^r(t) & \rho_{11}^r(t) \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned}
\rho_{00}^r(t) &= \frac{1}{2}\rho_{00}^r\left(1+e^{-4G^r(t)}\right)+\frac{1}{2}\rho_{11}^r\left(1-e^{-4G^r(t)}\right), \\
\rho_{01}^r(t) &= \frac{e^{-itE_J^r}}{2}\rho_{01}^r\left(1+e^{-4G^r(t)}\right)+\frac{1}{2}\rho_{10}^r\left(1-e^{-4G^r(t)}\right), \\
\rho_{01}^r(t) &= \frac{1}{2}\rho_{01}^r\left(1-e^{-4G^r(t)}\right)+\frac{e^{-itE_J^r}}{2}\rho_{10}^r\left(1+e^{-4G^r(t)}\right), \\
\rho_{11}^r(t) &= \frac{1}{2}\rho_{00}^r\left(1-e^{-4G^r(t)}\right)+\frac{1}{2}\rho_{11}^r\left(1+e^{-4G^r(t)}\right).
\end{aligned} \tag{19}$$

In the right hand side of Eq.(19) we denote $\rho_{ij}^r(0)$ with ρ_{ij}^r. If we do not consider the interactions between the qubits, the reduced density matrix of the combined system S becomes

$$\rho_s(t) = \rho^1(t)\otimes\rho^2(t)\,. \tag{20}$$

In the following, we set the system in a pure entangled state

$$\rho_s(0) = |\Psi\rangle\langle\Psi| \tag{21}$$

at $t=0$, where

$$|\Psi\rangle = \frac{1}{\sqrt{1+|\alpha|^2}}\left[|01\rangle + \alpha\,|10\rangle\right]. \tag{22}$$

The concurrence of the initial state is $C(0) = 2|\alpha|/\left(1+|\alpha|^2\right)$. At first, we investigate an especial case, namely, the case of $\alpha = 1$, which denotes the initial state is a maximum entangled state. We can easily obtain that after time t the density matrix of the open two-qubit system becomes

$$\rho_s(t) = \frac{1}{4}\begin{bmatrix} A & 0 & 0 & Ae^{-itE_J} \\ 0 & B & B & 0 \\ 0 & B & B & 0 \\ Ae^{itE_J} & 0 & 0 & A \end{bmatrix}. \tag{23}$$

where $A = 1 - e^{-4G^1(t)-4G^2(t)}$, and $B = 1 + e^{-4G^1(t)-4G^2(t)}$. Then, we can obtain the eigenvalues of the product $\rho_s(t)\,\tilde{\rho}_s(t)$ as

$$\mu_{1,2} = \frac{1}{4}\left(1 \pm 2e^{-4G^1(t)-4G^2(t)} + e^{-8G^1(t)-8G^2(t)}\right), \tag{24}$$

and

$$\mu_{3,4} = 0. \tag{25}$$

Finally, we can obtain the concurrence as

$$C(t) = \left|\sqrt{\mu_1}-\sqrt{\mu_2}\right| = e^{-4G^1(t)-4G^2(t)} = C(0)\delta^1(t)\,\delta^2(t)\,. \tag{26}$$

We call the $C(t)$ the decay of entanglement. The loss of the entanglement is $1-C(t)$. If we set $\alpha = -1$ or $\pm i$ we can obtain some similar results. It is very interesting that in this case the suppression factors describing the decoherence of the qubits and the entanglement of formation between the two qubits still has the compact relationship as Ref.[4, 5]. But, if the relationship is still preserved for other initial states? In the following we shall investigate the cases that the initial states are not the maximum entangled states. When $\alpha \neq \pm 1$ or $\pm i$ the operator calculations similar to above derivation becomes very complex, then we only be able to numerically investigate this problem with a concrete bath model. Suppose the two bathes with Ohmic noise spectrum are the same for convenience, which do not loss the universality. The spectral densities of the bathes may be expressed as

$$J(\omega) = \eta\omega e^{-\omega/\omega_c}.$$

It is well-known that when the summation in Eq.(13) is converted to the integration in the limit of infinite number of the bath modes, one has

$$G^r(t) = 2\eta\int d\omega e^{-\omega/\omega_c}\omega^{-1}\sin^2\frac{\omega t}{2}\coth\frac{\beta\omega}{2}$$

for the real $g(\omega)$.

$Fig.1$

In this paper we choose the cutoff frequency of the bath modes as $\omega_c = 10^{12}\ Hz$. It is pointed that the dimensionless strength of the dissipation is very small for some qubit environment. For example, for the Josephson charge qubit in Ohmic bath the value of η is about 10^{-6} [16, 17]. In our calculations, if we take $\eta = 10^{-6}$, the difference of the decay of concurrence $C(t)$ and the product of the suppression factors $\delta^1(t)\,\delta^2(t)$ with a initial concurrence $C(0)$ is very small. For convenience, where we denote $S(t) := C(0)\delta^1(t)\,\delta^2(t)$. When we increase η to 10^{-5} we can clearly see that the decay of concurrence decreases faster than $S(t)$ does. In Fig.1 we plot $C(t)$ versus time (with points), which is compared with $S(t)$ (with lines) in different initial states where (a) $\alpha = 1$, (b) $\alpha = 2$, and (c) $\alpha = 3$. It is shown that the concurrence is not equals to the product of suppression factors and its initial concurrence, namely $S(t)$ except for $\alpha = 1$. In plotting of the figure we choose a time unit, ks for convenience. Where we set $1\ ks = 1519.29\ ps$. From Fig.1 we see that the durative time of our calculations is $\tau = 12.15ps$, which is smaller than the characteristic time of the qubits [19]. So our calculations is accurate enough.

It is shown that system

$$\tilde{H} = \sum_{r=1}^{2}\left(\tilde{H}_s^r + H_B^r + \tilde{H}_I^r\right),$$

where

$$\begin{aligned}\tilde{H}_s^r &= -\frac{1}{2}E_J^r\sigma_z^r, H_B^r = \sum_k \omega_k^r b_k^{r\dagger} b_k^r,\\ \tilde{H}_I^r &= \sigma_x^r\sum_k\left(g_k^{r*}b_k^r + g_k^r b_k^{r\dagger}\right),\end{aligned}\tag{27}$$

is also a non-exactly solvable model and it has the same dynamics to system Eqs.(1-2) in short-time approximation [14, 19]. So the concurrence and the suppression factors describing the decoherence of the system may have a similar relationship to system Eqs.(1-2).

III. CONCLUSIONS

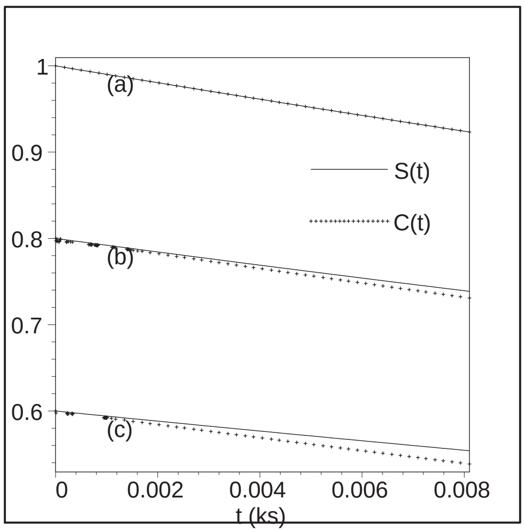
In this paper we investigated a non-exactly solvable open two-qubit model. We obtained that in a short time the decay of entanglement of formation is governed by the product of the suppression factors describing decoherence of the subsystems as the initial state of the two-qubit system is the maximum entangled state. This novel relationship is similar to the discover by V. Privman *et al.* in [4, 5], where the open two-qubit system is exactly solvable. Our work shows that when the initial state is not a maximum entangled state, after the open two-qubit evolve a short time t the entanglement of formation, concurrence decrease faster than the product of the suppression factors describing the decoherence of the two-qubit system, which is similar to the discover by Yu and Eberly in [9, 10]. It is shown that when the dissipation is not very weak (the dimensionless strength of the dissipation $\eta \gtrapprox 10^{-6}$) the entanglement is distinctly more frangible than the coherence for the quantum systems.

Acknowledgement 1 *The project was supported by National Natural Science Foundation of China (Grant No. 10347133) and Ningbo Youth Foundation (Grant No. 2004A620003).*

[1] M. A. Nielsen, and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Boston, 1995).
[4] D. Tolkunov, and V. Privman, Phys. Rev. A 71, 060308(R) (2005).
[5] D. Tolkunov, and V. Privman, Proc. SPIE 5815, 187 (2005).
[6] C. Bennett, D. DiVincenzo, J. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[7] M. J. Storcz, and F. K. Wilhelm, Phys. Rev. A 67, 042319 (2003).
[8] B. Ischi, M. Hilke, and M. Dube, Phys. Rev. B 71, 195325 (2005).
[9] T. Yu, and J. H. Eberly, Phys. Rev. B 66, 193306 (2002); *ibid.* 68, 165322 (2003).
[10] T. Yu, and J. H. Eberly, Phys. Rev. lett. 93. 140404 (2004).
[11] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987); 67, 725(E) (1995).
[12] S. K. Gray, and J. M. Verosky, J. Chem. Phys. 99, 8680 (1993).
[13] S. Blanes, and P. C. Moan, Phys. Lett. A 265, 35 (2000).
[14] L. Fedichkin, A. Fedorov, and V. Privman, Phys. Lett. A 328, 87 (2004); A. Fedorov, L. Fedichkin and V. Privman, J. Comp. Theor. Nanosci. 1, 132 (2004); D. Tolkunov and V. Privman, Phys. Rev. A 69, 062309 (2004); V. Privman, J. Stat. Phys. 110, 957 (2003).
[15] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998); S. Hill, and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[16] M. Goernale, M. Grifoni, and G. Schön, Chem. Phys. 268, 273 (2001).
[17] Y. Makhlin, G. Schön, and A. Shnirman, Chem. Phys. 296, 315 (2003).
[18] C. P. Sun, S. Yi, and L. You, Phys. Rev. A 67, 063815 (2003).
[19] X. T. Liang, and Y. J. Xiong, Physica B 362, 243 (2005).

IV. CAPTION OF FIG.1

Fig.1: The concurrence $C(\alpha, t)$ (points) and $S(\alpha, t) = 2\alpha/(1+|\alpha|^2)\delta^1(t)\,\delta^2(t)$ (lines) of a two-qubit system in their Ohmic bathes for different initial states (a) $\alpha = 1$, (b) $\alpha = 2$, (c) $\alpha = 3$. Here we take dimensionless strength of the dissipation $\eta = 1 \times 10^{-5}$, the cutoff frequencies of the bath modes $\omega_c = 10^{12}\ Hz$.


1
0.9
0.8
0.7
0.6
(a)
(b)
(c)
S(t)
C(t)
0
0.002
0.004
0.006
0.008
t (ks)